Working with Henry Wynn: the Quest for Orderings

Alessandra Giovagnoli
(previously) University of Bologna

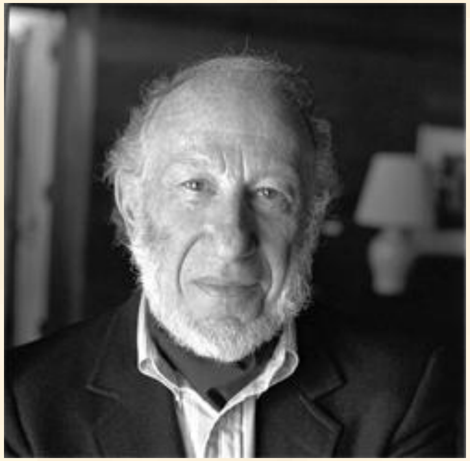
from:

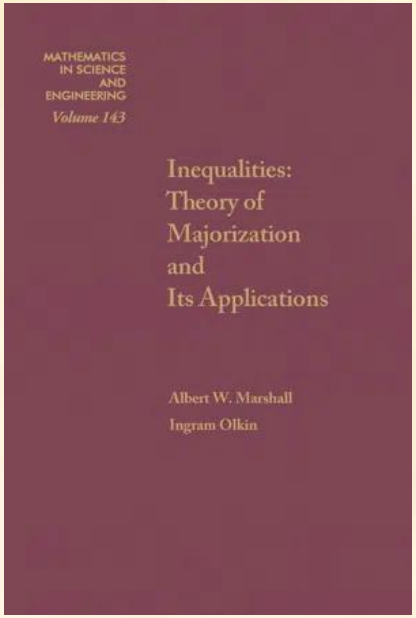
A. Giovagnoli, H. Maruri-Aguilar, L. Pronzato, E. Riccomagno, A. Zhigljavsky
“Working with Henry Wynn: a challenge and an exceptional opportunity”
(to appear)

Imperial College, London, 1976-77

Motto

"SCIENTIFIC KNOWLEDGE, THE CROWNING GLORY AND THE SAFEGUARD OF THE EMPIRE"



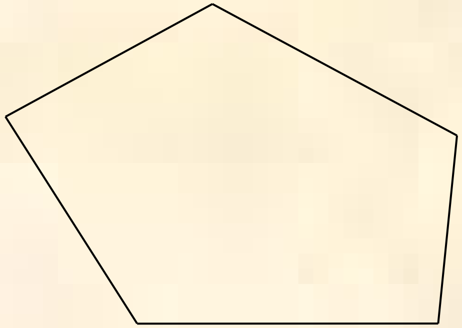


First edition, 1979

Majorization $\prec$

$\boldsymbol{x}, \boldsymbol{y}$ vectors in $\mathbb{R}^n$

$\boldsymbol{x} \prec \boldsymbol{y}$ *means* $\boldsymbol{x}$ *is in the* convex hull *of all the vectors obtained through permutations of the entries of* $\boldsymbol{y}$



The entries of vector $\boldsymbol{x}$ are less spread out than those of $\boldsymbol{y}$

A more common definition of $\prec$

$$x_{[1]} \geq x_{[2]} \geq \ldots\ldots \geq x_{[n]}$$

$$y_{[1]} \geq y_{[2]} \geq \ldots\ldots \geq y_{[n]}$$

$$\boldsymbol{x} \prec \boldsymbol{y} \iff \begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} & k = 1, \ldots., n-1 \\ \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \end{cases}$$

- Majorization is a pre-ordering
- Order-preserving functions are called **Schur-convex**
  (e.g. any permutation invariant convex fn, and many more)

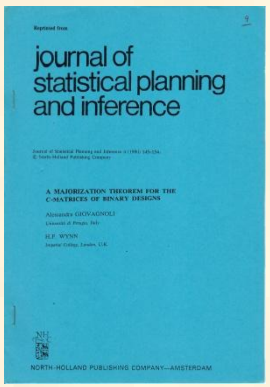
Optimal Designs

Jack Kiefer **1924-1981**

Cornell and Berkeley, U.S.A

Criteria for Linear Model

- A-optimality
- D-optimality
- E-optimality
- G-optimality



First joint paper with Henry:

- "A **majorization** theorem for the C-matrices of binary **designs**" (1980)

The appropriate design ordering is $\prec$ on eigenvalues of information matrix

- "**Optimum** continuous block **designs**" *Proc. Royal Soc. London* (1981)
- "**Schur-optimal** block **designs** for treatments with a control" In: *Proceedings of the Berkeley Conference in honor of Jerzy Neyman and Jack Kiefer* Wadsworth,CA, (1985)

Weak majorizations

(lower $\prec_w$ & upper $\prec^w$)

$$x \prec_w y \Leftrightarrow \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$$

$$x \prec^w y \Leftrightarrow \sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}$$

increasing rearrangement

Equivalent definitions of $\prec_w$ and $\prec^w$ combining $\prec$ with entrywise ordering $\leq$ (after rearrangement)

$\boldsymbol{x} \prec_w \boldsymbol{y} \quad \Leftrightarrow \quad \exists\ \boldsymbol{u}$ in $\mathbb{R}^d$ $\quad \boldsymbol{x} \prec \boldsymbol{u}$ and $\boldsymbol{u} \leq \boldsymbol{y}$

$\boldsymbol{x} \prec^w \boldsymbol{y} \quad \Leftrightarrow \quad \exists\ \boldsymbol{v} \quad \boldsymbol{x} \prec \boldsymbol{v}$ and $\boldsymbol{v} \geq \boldsymbol{y}$

The appropriate design ordering is $\prec^w$ on eigenvalues of information matrix

Order relations in Statistics

Axiomatic approach to statistical concepts by means of **equivalence** and **order** relations *Bickel and Lehmann* (1975, 1976, 1979)

transitive: $x <_{\Delta} y$ AND $y <_{\Delta} z \Rightarrow x <_{\Delta} z$ $\forall$ x, y, z in *S*

(pre-order)

A large parte of statistics is concerned with indicators: **precise measurements of vague concepts**

Approach through orderings allows **better investigation of statistical concepts**

Properties of statistical indicators in many instances follow from properties of the underlying order relations

Imperial College, 1983: *GW-majorization*

Giovagnoli, Wynn

- "G-majorization with applications to matrix **orderings**" *Linear Algebra Appl*. 1985

G-MAJORIZATION $\quad \boldsymbol{x} \prec_{\mathcal{G}} \boldsymbol{y} \quad \Leftrightarrow \quad \boldsymbol{x} = \sum_{\mathrm{i}} \alpha_i g_i(\boldsymbol{y}) \; \alpha_i \geq 0, \; \sum_{\mathrm{i}} \alpha_i = 1,$

$\mathcal{G}$ any linear group on $\mathbb{R}^{\mathrm{d}}$

GW-MAJORIZATION G *for* group, W *for* weak (upper & lower)

Weak G-majorization: $\quad \triangleleft \quad$ any $\mathcal{G}$-compatible order in $\mathbb{R}^{\mathrm{d}}$

$$\boldsymbol{x} \triangleleft \boldsymbol{y} \quad \Rightarrow \quad \sum_{\mathrm{i}} \alpha_i g_i(\boldsymbol{x}) \triangleleft \sum_{\mathrm{i}} \alpha_i g_i(\boldsymbol{y})$$

$$\boldsymbol{x} \prec_{\mathrm{GW}} \boldsymbol{y} \quad \Leftrightarrow \quad \exists \; \boldsymbol{u} \quad \boldsymbol{x} \prec_{\mathrm{G}} \boldsymbol{u} \text{ and } \boldsymbol{u} \triangleleft \boldsymbol{y}$$

$$\boldsymbol{x} \prec^{\mathrm{GW}} \boldsymbol{y} \quad \Leftrightarrow \quad \exists \; \boldsymbol{v} \quad \boldsymbol{x} \prec^{\mathrm{G}} \boldsymbol{v} \text{ and } \boldsymbol{y} \triangleleft \boldsymbol{v}$$

GW applied to designs

The 3 authors, Tashkent, 1986

- **Giovagnoli, Pukelsheim, Wynn**

  "Group invariant **orderings** and experimental **designs**"

  J. Statist. Planning Inf. (1987)

A e B symmetric non-negative definite matrices , with $\leq_L$ *Loewner ordering* :

$\mathbb{G}$ *= a subgroup of orthogonal matrices acting by congruence*

The appropriate design ordering is $\prec^{GW}$ on information matrices

$M_1 \prec^{GW} M_2$ means M_1 <u>more informative</u> than M_2

Known optimality criteria are consistent with $\prec^{GW}$ **(are order-preserving).**

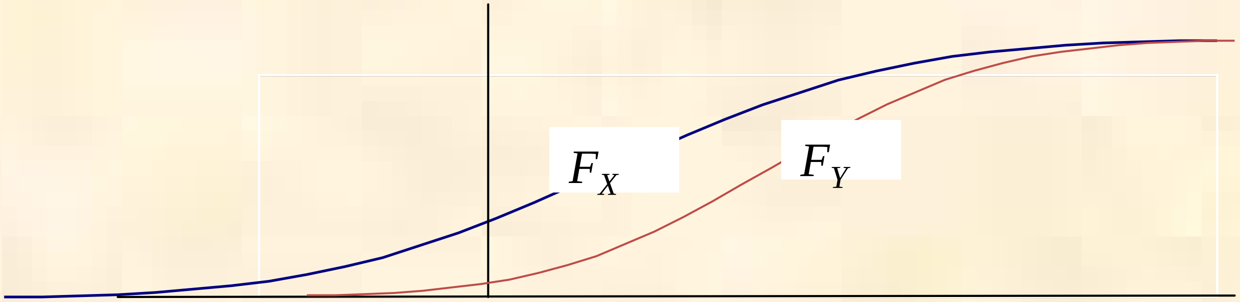
- Special case: if $\mathbb{G}$ = the group of **all** orthogonal matrices, then $\prec^{GW}$ becomes upper weak majorization of eigenvalues of M.

Stochastic Orderings

Comparing random variables X, Y

For a start, comparing quantity, location, size, length...

First degree Stochastic Dominance : $X <^{FSD} Y$ **iff** $F_X(x) \geq F_Y(x)$



Other possible comparisons

Variability, Dispersion, Concentration both univariate and multivariate

Association, Dependence, Concordance, ***Agreement*** multivariate

Statistical indicators are order-preserving functions w.r.t. some stochastic order relation

Giovagnoli, Wynn

- "Multivariate dispersion **orderings**" *Statistics and Probability Letters* (1995)

Stochastic Orderings: the discrete case

Henry at **I.S.A., University of Bologna, 2005**

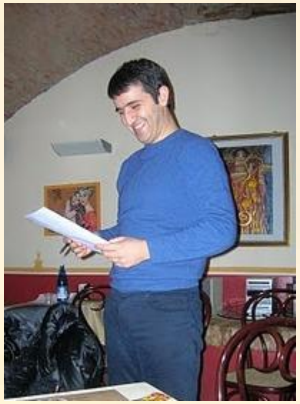
Giovagnoli, **Wynn**

- “Stochastic **orderings** for <u>discrete</u> random variables” *Statistics and Probability Letters* (2008)

Dependence Orderings in the discrete case

Giovagnoli, **Marzialetti**, **Wynn:**

- "Bivariate dependence **orderings** for unordered categorical variables" in *Optimal Design and Related Areas in Optimization and Statistics,* L. Pronzato and A.A. Zhigljavsky, eds., Springer (2009)



Johnny Marzialetti

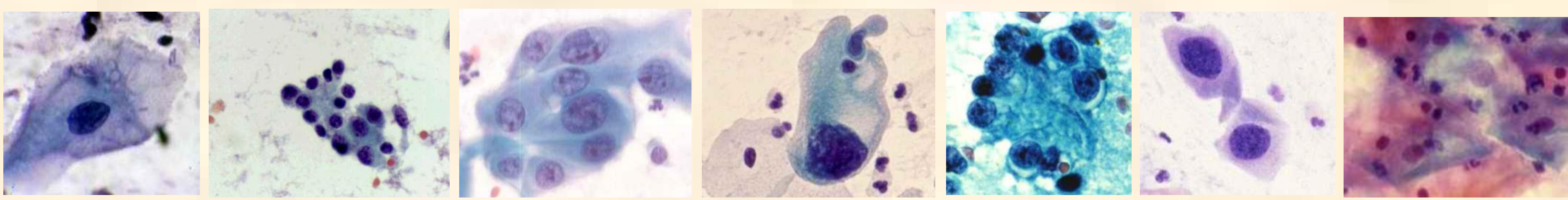
Celebrating Henry's 60th Birthday, Juan-Les-Pins
26 May 2005

Giovagnoli, **Marzialetti**, **Wynn:**

- "An approach to inter-rater agreement through stochastic **orderings**: the discrete case" *Metrika* (2008)

From biopsy to diagnosis

Maria Rosaria Giovagnoli
Head, Cytopathology Lab



Comparing the diagnostic agreement among different groups of observers on the same sample (smears of cytologycal cells)

Agreement: An Axiomatic Approach

<u>Agreement between independent observers</u> (people or diagnostic instruments) when rating the same subject, e.g. a patient on seriousness of illness

Scores x_i **on a continuous or discrete ordered scale** I

$(x_1,\ldots,x_d)$ = ratings by group of d observers $\qquad S = \underbrace{I\times I\times\ldots\times I}_{d \text{ times}}$

A.1 (*Perfect agreement*)

If all observers give same score $(x,x,\ldots,x)$, agreement is a maximum

A.2 (*Equivalence*)

If order of observers is permuted, agreement stays the same

A.3 (*Invariance*) Agreement does not change under the same shift of all the scores $(x_1 + a, x_2 + a,\ldots, x_d + a)$

A.4 (*Compromise*) If two observers accept to make their scores more similar

$$(x_1,\ldots, x_d) \rightarrow (x_1 - \Delta, x_2 + \Delta, x_3,\ldots, x_d)$$

then agreement increases

Stochastic agreement ordering for multivariate r.v. on an ordered scale

"Agreement ordering" $<_A$ on $S = \boldsymbol{I}^d$ is <u>reverse</u> of $\mathcal{G}$-*majorization*

where $\mathcal{G}$ = {permutation of coordinates + shifts}

$$\boldsymbol{x} <_A \boldsymbol{y} \quad \Leftrightarrow \quad (y_1 - \bar{y}, y_2 - \bar{y}, \ldots, y_d - \bar{y}) \prec (x_1 - \bar{x}, x_2 - \bar{x}, \ldots, x_d - \bar{x}) \qquad \bar{x} = \frac{1}{d}\sum_{i=1}^{d} x_i$$

i.e. $\boldsymbol{y}$ is more concentrated around the average score than $\boldsymbol{x}$

In the case of two observers, $<_A$ is a total ordering.

d **observers rating** n **subjects →** random vector of scores $\boldsymbol{X} = (X_1,\ldots,X_d)$

From $<_A$ on S to **stochastic agreement** $<_A^{st}$ of multivariate $\boldsymbol{X}, \boldsymbol{Y}$

$$\boldsymbol{X} <_A^{st} \boldsymbol{Y} \quad \Leftrightarrow \quad \Pr\{\boldsymbol{x} <_A \boldsymbol{X}\} \leq \Pr\{\boldsymbol{x} <_A \boldsymbol{Y}\} \qquad \forall\, \boldsymbol{x} \in S$$

Example: d = 2 observers, m = 4 possible scores

		Observer 2				
		0	1	2	3	
Observer 1	0	p_{00}	p_{01}	p_{02}	p_{03}	p_{0+}
	1	p_{10}	p_{11}	p_{12}	p_{13}	p_{1+}
	2	p_{20}	p_{21}	p_{22}	p_{23}	p_{2+}
	3	p_{30}	p_{31}	p_{32}	p_{33}	p_{3+}
		p_{+0}	p_{+1}	p_{+2}	p_{+3}	1

$\boldsymbol{X}$ *is bivariate r.v.*

$P = [\, p_{ij} \,] \qquad i, j = 0, 1, 2, 3$

probability distribution of pairs of ratings

Also $\boldsymbol{Y} \qquad Q = [\, q_{ij} \,]$

$$tr_0 P = p_{00} + p_{11} + p_{22} + p_{33}$$
$$tr_1 P = p_{01} + p_{12} + p_{23} + p_{10} + p_{21} + p_{23}$$
$$tr_2 P = p_{02} + p_{13} + p_{20} + p_{31}$$
$$tr_3 P = p_{03} + p_{30}$$

$$\boxed{P \geq_A^{st} Q \quad \Leftrightarrow \quad \begin{cases} tr_0 P \geq tr_0 Q \\ tr_0 P + tr_1 P \geq tr_0 Q + tr_1 Q \\ tr_0 P + tr_1 P + tr_2 P \geq tr_0 Q + tr_1 Q + tr_2 Q \end{cases}}$$

Measures of agreement

Order-preserving functions w.r.t $<^{st}_{\mathrm{A}}$

(Weighted) Total Proportion of Agreement **TPA**$_w$

$$\sum_{i=1}^{m}\sum_{j=1}^{m} w_{ij}p_{ij}$$

with weights $0 \leq w_{ij} \leq 1$ $\quad w_{ij} = w_{ji}$ and $w_{ii} = 1$

Commonly used weights: $w_{ij} = 1 - \dfrac{|i-j|}{(m-1)}$ or $w_{ij} = 1 - \dfrac{(i-j)^2}{(m-1)^2}$

In medical literature

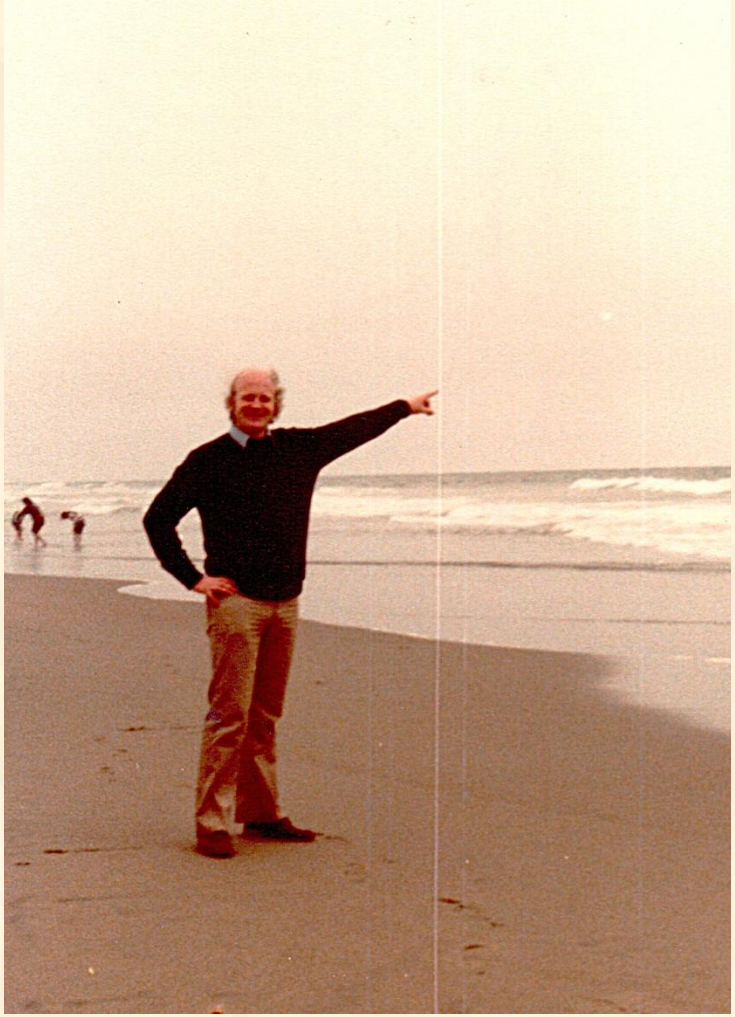
Cohen's weighted Kappa

?

$$\boxed{\kappa_w = \frac{\sum_{i=1}^{m}\sum_{j=1}^{m} w_{ij}\cdot p_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{m} w_{ij}\cdot p_{i+}\cdot p_{+j}}{1 - \sum_{i=1}^{m}\sum_{j=1}^{m} w_{ij}\cdot p_{i+}\cdot p_{+j}}}$$

(When $w_{ij} = 0$ for $i \neq j$, classic Cohen's Kappa κ)

The Future…



Henry, 1981, California